

ON SCALABLE MONOIDS

DAN JONSSON

ABSTRACT. This brief exposition presents some basic properties of scalable monoids and quantity spaces, introduced in [Jon14a].

1. INTRODUCTION

Let X be a non-empty set and R a commutative unital ring. An associative unital algebra with X as carrier set combines three operations on X :

- (1) *addition* of elements of X , a binary operation $+$: $X \times X \rightarrow X$, written $(x, y) \mapsto x + y$, such that $(X, +)$ is an abelian group;
- (2) *multiplication* of elements of X , a binary operation \cdot : $X \times X \rightarrow X$, written $(x, y) \mapsto x \cdot y$ or $(x, y) \mapsto xy$, such that (X, \cdot) is a monoid;
- (3) *scalar multiplication* of elements of X by elements of R , a monoid action $R \times X \rightarrow X$, written $(\alpha, x) \mapsto \alpha x$, where the multiplicative monoid of R acts on X , so that $1x = x$ and $\alpha(\beta x) = (\alpha\beta)x$.

These structures are linked pairwise:

- (a) addition and multiplication are linked by the distributive laws $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$; one of these laws suffices when multiplication in X is commutative;
- (b) addition and scalar multiplication are linked by the distributive laws $\alpha(x + y) = \alpha x + \alpha y$ and $(\alpha + \beta)x = \alpha x + \beta x$;
- (c) multiplication and scalar multiplication are linked by the laws $\alpha(xy) = (\alpha x)y$ and $\alpha(xy) = x(\alpha y)$; one of these laws suffices when multiplication in X is commutative.

Related algebraic systems can be obtained from associative unital algebras by removing one of the three operations and hence the links between the removed operation and the two others. Two cases are very familiar. A unital *ring* has only addition and multiplication of elements, linked as described in (a). A *module* has only addition and scalar multiplication of elements, linked as described in (b). The question arises whether it makes sense to define an “algebra without an additive group”, with only multiplication and scalar multiplication, linked as described in (c). Would such a notion be of mathematical interest and useful for applications? Some reasons to give an affirmative answer to that question are found in this article.

2. BASIC NOTIONS

It is tempting to call an “algebra without an additive group” an “alebra”, but that may sound somewhat frivolous, and considering the nature of this algebraic structure, “scalable monoid” [Jon14a], abbreviated “scaloid”, might be a better name. This structure can be defined more or less broadly. In the usual definition of an

algebra X over a ring R , multiplication in R is assumed to be commutative but multiplication in X is not. The following definition is on the same level of generality.

Definition 1. A *scalable monoid*, or *scaloid*, is an algebraic structure incorporating structures described by (2), (3) and (c) above.

More explicitly, a scalable monoid X over a commutative unital ring R is a monoid X such that there is a function

$$\sigma : R \times X \rightarrow X, \quad (\alpha, x) \mapsto \alpha x,$$

called *scalar multiplication*, such that for any $\alpha, \beta \in R$ and any $x, y \in X$ we have

- (1) $1x = x$,
- (2) $\alpha(\beta x) = (\alpha\beta)x$,
- (3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$.

For clarity, we may denote the unit element of the monoid X by 1_X or $\mathbf{1}$ and the unit element of the ring R by 1_R . \square

An *invertible* element of a scaloid X is an element $x \in X$ which has an *inverse* $x^{-1} \in X$ such that $xx^{-1} = x^{-1}x = 1_X$. We can define positive powers of x , denoted x^k , in the usual way; if x is invertible, negative powers of x can be defined by setting $x^k = (x^{-1})^{-k}$. By convention, $x^0 = 1_X$. By associativity, $x^k x^\ell = x^{(k+\ell)}$.

The following two simple facts about scaloids will be used repeatedly:

Claim 1. Let X be a scalable monoid over R .

- (1) $(\alpha x)(\beta y) = (\alpha\beta)(xy)$ for any $x, y \in X$ and $\alpha, \beta \in R$.
- (2) $\alpha(\beta x) = \beta(\alpha x)$ and $(\alpha\beta)x = (\beta\alpha)x$ for any $x \in X$ and $\alpha, \beta \in R$.

Proof. (1). $(\alpha x)(\beta y) = \alpha(x(\beta y)) = \alpha(\beta(xy)) = (\alpha\beta)(xy)$.
 (2). $\alpha(\beta x) = \alpha(\beta(\mathbf{1}x)) = \alpha((\beta\mathbf{1})x) = (\beta\mathbf{1})(\alpha x) = \beta(\mathbf{1}(\alpha x)) = \beta(\alpha(\mathbf{1}x)) = \beta(\alpha x)$, so $(\alpha\beta)x = (\beta\alpha)x$. \square

Remark. In the proofs given below, only the fact that $\alpha\beta$ and $\beta\alpha$ act in the same way on X is used, and this follows from (2) and (3) in Definition 1, so the stronger assumption in Definition 1 that R is commutative is not really needed here.

3. REALMS AND ADDITIVE GROUP OPERATIONS

By removing the additive group of an algebra, we remove the three group operations $(x, y) \mapsto x + y$, $(x) \mapsto -x$ and $() \mapsto \mathbf{0}$ (addition, taking the inverse and selecting the group identity). It turns out, however, that under natural assumptions the group structure can be partially recovered. Specifically, a scalable monoid over R can be partitioned into “realms” within each of which a group structure corresponding to the additive group structure in R can often be defined.

3.1. Commensurability and realms.

Definition 2. Let X be a scalable monoid over R , and let \sim be a relation on X such that $x \sim y$ if and only if $\alpha x = \beta y$ for some $\alpha, \beta \in R$. Then x and y are said to be *commensurable*, and the relation \sim is called *commensurability*. \square

As a trivial consequence of this definition, $x \sim \lambda x$ for any $x \in X$ and $\lambda \in R$, since $\lambda x = 1(\lambda x)$.

Proposition 1. *Commensurability is an equivalence relation.*

Proof. The commensurability relation \sim is reflexive because $1x = 1x$, it is symmetric because if $\alpha x = \beta y$ then $\beta y = \alpha x$, and it is transitive because if $\alpha x = \beta y$ and $\gamma y = \delta z$ then $\gamma(\alpha x) = \gamma(\beta y)$ and $\beta(\gamma y) = \beta(\delta z)$, so $\gamma(\beta y) = \beta(\gamma y)$ implies $(\gamma\alpha)x = (\beta\delta)z$. \square

Definition 3. Let X be a scalable monoid. A (*commensurability*) *realm* in X is an equivalence class with respect to commensurability. We denote the realm containing $x \in X$ by $[x]$. \square

Proposition 2. Let X be a scalable monoid. The commensurability relation \sim is a congruence relation on X , so the corresponding set of equivalence classes can be made into an algebraic structure X/\sim with a binary operation $([x], [y]) \mapsto [x][y]$ defined by

$$[x][y] = [xy]$$

for any $x, y \in X$. X/\sim is a monoid with $[1]$ as unit element.

Proof. If $\alpha x = \alpha' x'$ and $\beta y = \beta' y'$ then $(\alpha x)(\beta y) = (\alpha' x')(\beta' y')$, so $(\alpha\beta)(xy) = (\alpha'\beta')(x'y')$. Thus, if $x \sim x'$ and $y \sim y'$ then $xy \sim x'y'$, so the product in X/\sim given by $[x][y] = [xy]$ is well-defined.

X/\sim is the image of X under the mapping $h : x \mapsto [x]$, and as $[x][y] = [xy]$ and $1x = x1 = x$, h is a homomorphism of monoids, meaning that X/\sim is a monoid with $h(1) = [1]$ as unit element. \square

Note that a scaloid is akin to a tensor product of modules. Recall that the tensor product of $u \in U$ and $v \in V$ is a tensor $u \otimes v$ belonging to $U \otimes V$; this is a module in general distinct from U and V . Similarly, the product xy of x and y in a scaloid X does not in general belong to the same realm as x or y ; in general $[x], [y] \neq [xy]$. This suggests an application of scaloids: “quantities” such as 1 N , 2 m or 2 Nm can be regarded as elements of a scaloid, specifically quantities with different “sorts” or “dimensions”, corresponding to different equivalence classes of such quantities, namely commensurability realms.

Pursuing the analogy with tensor products, certain additional assumptions allow us to regard each realm as a module, making it possible to add and subtract quantities as described below.

3.2. Recovering additive group operations in realms.

Definition 4. Let X be a scalable monoid over R , and consider a realm $\mathfrak{R} \subset X$. A *pivot* for \mathfrak{R} is an element $p \in \mathfrak{R}$ such that for every $x \in \mathfrak{R}$ there is a unique $\lambda \in R$ such that $x = \lambda p$. \square

Proposition 3. If p and p' are pivots for \mathfrak{R} , so that $x = \alpha p = \alpha' p'$ and $y = \beta p = \beta' p'$ for any $x, y \in \mathfrak{R}$, then $(\alpha + \beta)p = (\alpha' + \beta')p'$.

Proof. As $p' \in \mathfrak{R}$, there is a unique $\lambda \in R$ such that $p' = \lambda p$. Thus, $(\alpha' + \beta')p' = (\alpha' + \beta')\lambda p = (\alpha'\lambda + \beta'\lambda)p$. Also, $x = \alpha p = \alpha' p' = \alpha'\lambda p$ implies $\alpha = \alpha'\lambda$, and $y = \beta p = \beta' p' = \beta'\lambda p$ implies $\beta = \beta'\lambda$. \square

Hence, the sum of two elements of a scalable monoid can be defined as follows:

Definition 5. Let X be a scalable monoid over R , and let p be a pivot for a realm \mathfrak{R} . If $x = \alpha p$ and $y = \beta p$, we set

$$x + y = (\alpha + \beta)p. \quad \square$$

Note that if $x = \alpha p$ and $y = \beta p$ then $\beta x = \beta(\alpha p) = \alpha(\beta p) = \alpha y$, so $x + y$ can be defined *only if* $x \sim y$ – one cannot add “incommensurable” quantities.

As a trivial consequence of this definition, addition in a realm is commutative.

If p is a pivot for a realm \mathfrak{R} and $p' \in \mathfrak{R}$ then $0p' = 0(\lambda p) = 0p$, so there is a unique element $\mathbf{0}_{\mathfrak{R}} \in \mathfrak{R}$ defined by $\mathbf{0}_{\mathfrak{R}} = 0p$ for any pivot p for \mathfrak{R} , and it follows immediately from Definition 5 that

$$x + \mathbf{0}_{\mathfrak{R}} = \mathbf{0}_{\mathfrak{R}} + x = x$$

for any $x \in \mathfrak{R}$. We also set

$$-x = (-1)x \quad \text{and} \quad y - x = y + (-x)$$

for any $x, y \in \mathfrak{R}$, so that

$$x - x = -x + x = \mathbf{0}_{\mathfrak{R}}$$

for any $x \in \mathfrak{R}$.

4. FREE COMMUTATIVE SCALABLE MONOIDS

A commutative scalable monoid X is one where $xy = yx$ for all $x, y \in X$. In this section, only commutative scalable monoids will be considered.

Definition 6. Let X be a commutative scalable monoid over R . A (finite) *basis* for X is a set $B = \{b_1, \dots, b_n\}$ of invertible elements of X such that every $x \in X$ has a unique (up to order of factors) expansion

$$x = \mu \prod_{i=1}^n b_i^{k_i},$$

where $\mu \in K$ and k_1, \dots, k_n are integers. \square

Any product of invertible quantities is invertible, so any product of basis elements is invertible. As X is commutative,

$$\left(\mu \prod_{i=1}^n b_i^{k_i} \right) \left(\nu \prod_{i=1}^n b_i^{\ell_i} \right) = (\mu\nu) \prod_{i=1}^n b_i^{(k_i + \ell_i)}.$$

4.1. Quantity spaces. Below, only scaloids over a field K will be considered.

Definition 7. A (finite-dimensional) *free commutative scalable monoid over a field K* , or a *quantity space* over K , is a commutative scalable monoid over K which has a (finite) basis. \square

With a view to applications, elements of a quantity space may be called *quantities*, and realms in a quantity space may be called *dimensions*. As explained in [Jon14a], a basis for a quantity space can be interpreted as a *system of fundamental units of measurement*.

Claim 2. Let X be a quantity space over K with a basis $\{b_1, \dots, b_n\}$, and consider $x = \mu \prod_{i=1}^n b_i^{k_i}$ and $y = \nu \prod_{i=1}^n b_i^{\ell_i}$. The following conditions are equivalent:

- (1) $x \sim y$.
- (2) $k^i = \ell^i$ for $i = 1, \dots, n$.
- (3) $\nu x = \mu y$.

Proof. (1) \Rightarrow (2). If $x \sim y$ then $(\alpha\mu) \prod_{i=1}^n b_i^{k_i} = z = (\beta\nu) \prod_{i=1}^n b_i^{\ell_i}$ for some $\alpha, \beta \in K$. As the expansion of z is unique, $k^i = \ell^i$ for $i = 1, \dots, n$.

The implications (2) \Rightarrow (3) and (3) \Rightarrow (1) are trivial. \square

4.2. The measure of a quantity.

Definition 8. Let X be a quantity space over K , and let $B = \{b_1, \dots, b_n\}$ be a basis for X . The uniquely determined scalar $\mu \in K$ in the expansion

$$x = \mu \prod_{i=1}^n b_i^{k_i}$$

is called the *measure* of x relative to B and will be denoted by $\mu_B(x)$. \square

For example, $\mu_B(\mathbf{1}) = 1$ for any B , because $\mathbf{1} = 1 \cdot \mathbf{1} = 1 \prod_{i=1}^n b_i^0$ for any B . Note that the identity $\nu x = \mu y$ in Claim 2 can be written as $\mu_B(y) x = \mu_B(x) y$.

Relative to a fixed basis, measures of quantities can be used as proxies for the quantities themselves. For example, the measure of a product of quantities is equal to the product of the measures of these quantities.

Proposition 4. Let $B = \{b_1, \dots, b_n\}$ be a basis for a quantity space X over K .

- (1) For any $x \in X$ and $\lambda \in K$, $\mu_B(\lambda x) = \lambda \mu_B(x)$.
- (2) For any $x, y \in X$, $\mu_B(xy) = \mu_B(x) \mu_B(y)$.

Proof. We have $x = \mu \prod_{i=1}^n b_i^{k_i}$ and $y = \nu \prod_{i=1}^n b_i^{\ell_i}$, where $\mu, \nu \in K$.

- (1). $\lambda x = \lambda \mu \prod_{i=1}^n b_i^{k_i}$, so $\mu_B(\lambda x) = \lambda \mu = \lambda \mu_B(x)$.
- (2). $(\mu \prod_{i=1}^n b_i^{k_i}) (\nu \prod_{i=1}^n b_i^{\ell_i}) = (\mu \nu) \prod_{i=1}^n b_i^{(k_i + \ell_i)}$, so $\mu_B(xy) = \mu \nu = \mu_B(x) \mu_B(y)$.

\square

Proposition 5. A quantity $x \in X$ is invertible if and only if $\mu_B(x) \neq 0$, and for any invertible $x \in X$ we have $\mu_B(x^{-1}) = \mu_B(x)^{-1} = 1/\mu_B(x)$.

Proof. If $\mu_B(x) \neq 0$ and $x = \mu_B(x) \prod_{i=1}^n b_i^{k_i}$ then

$$\frac{\prod_{i=1}^n b_i^{-k_i}}{\mu_B(x)} x = x \frac{\prod_{i=1}^n b_i^{-k_i}}{\mu_B(x)} = \left(\mu_B(x) \prod_{i=1}^n b_i^{k_i} \right) \frac{\prod_{i=1}^n b_i^{-k_i}}{\mu_B(x)} = \mathbf{1},$$

so x is invertible. If, conversely, x has an inverse x^{-1} then $\mu_B(x) \mu_B(x^{-1}) = \mu_B(xx^{-1}) = \mu_B(\mathbf{1}) = 1$, so $\mu_B(x) \neq 0$ and $\mu_B(x^{-1}) = 1/\mu_B(x)$. \square

Proposition 6. Let X be a quantity space over K . For every $x \in [\mathbf{1}]$, $\mu_B(x)$ does not depend on B .

Proof. Let $B = \{b_1, \dots, b_n\}$ and $\widehat{B} = \{\widehat{b}_1, \dots, \widehat{b}_m\}$ be bases for X . The quantity $\mathbf{1}$ has the expansions $\mathbf{1} = 1 \prod_{i=1}^n b_i^0$ and $\mathbf{1} = 1 \prod_{i=1}^m \widehat{b}_i^0$. In view of Claim 2, therefore, $x = \mu_B(x) \prod_{i=1}^n b_i^0$ and $x = \mu_{\widehat{B}}(x) \prod_{i=1}^m \widehat{b}_i^0 = \mu_{\widehat{B}}(x) \cdot \mathbf{1} = \mu_{\widehat{B}}(x) \prod_{i=1}^n b_i^0$, so $\mu_B(x) = \mu_{\widehat{B}}(x)$. \square

It is common to refer to a quantity $x \in [\mathbf{1}]$ as a “dimensionless quantity”, although x is not really “dimensionless” – it belongs to, or “has”, the dimension $[\mathbf{1}]$. The fact that the measure of any $x \in [\mathbf{1}]$ does not depend on a choice of basis – that is, a choice of fundamental units of measurement – is the foundation for dimensional analysis [Bri22, Bar96, Jon14a].

4.3. Additive group operations in quantity spaces.

Proposition 7. *Let X be a quantity space over K .*

- (1) *If $x, y \in X$, $x \sim y$ and x is invertible then $y = \kappa x$ for some $\kappa \in K$.*
- (2) *If $x \in X$ is invertible and $\lambda x = \lambda' x$ for some $\lambda, \lambda' \in K$ then $\lambda = \lambda'$.*

Proof. (1). $\mu_B(y)x = \mu_B(x)y$ by Claim 2, and $\mu_B(x) \neq 0$ by Proposition 5,, so

$$y = \frac{\mu_B(y)}{\mu_B(x)}x.$$

(2). Let x have the expansion $x = \mu \prod_{i=1}^n b_i^{k_i}$ relative to a basis $\{b_1, \dots, b_n\}$ for X . If $\lambda x = \lambda' x$ then $\lambda \mu \prod_{i=1}^n b_i^{k_i} = z = \lambda' \mu \prod_{i=1}^n b_i^{k_i}$, so $\lambda \mu = \lambda' \mu$ since the expansion of z is unique, and $\mu \neq 0$ since x is invertible, so $\lambda = \lambda'$. \square

Thus, every invertible quantity is a pivot for the realm to which it belongs. This has an important consequence:

Corollary 1. *Every realm in a quantity space has a pivot.*

Proof. An equivalence class is not empty, so for every realm \mathfrak{R} there is some quantity $\mu \prod_{i=1}^n b_i^{k_i} \in \mathfrak{R}$, so there is an invertible element $1 \prod_{i=1}^n b_i^{k_i} \in \mathfrak{R}$. \square

In a quantity space, $x + y$ is thus defined *if and only if* $x \sim y$. Addition of quantities within a realm connects with addition of measures in the following way:

Proposition 8. *Let X be a quantity space over K . For any basis B for X and $x, y \in X$ such that $x \sim y$, we have $\mu_B(x) + \mu_B(y) = \mu_B(x + y)$.*

Proof. Let $x = \mu_B(x) \prod_{i=1}^n b_i^{k_i}$ and $y = \mu_B(y) \prod_{i=1}^n b_i^{k_i}$ be the expansions of x and y relative to $B = \{b_1, \dots, b_n\}$. $\prod_{i=1}^n b_i^{k_i}$ is invertible, and thus a pivot, so

$$x + y = \mu_B(x) \prod_{i=1}^n b_i^{k_i} + \mu_B(y) \prod_{i=1}^n b_i^{k_i} = (\mu_B(x) + \mu_B(y)) \prod_{i=1}^n b_i^{k_i},$$

so we have obtained the unique expansion of $x + y$ relative to B , and this expansion shows that $\mu_B(x + y) = \mu_B(x) + \mu_B(y)$. \square

In words, the measure of a sum of quantities is equal to the sum of the measures of these quantities. Also, $\mu_B(\lambda x) = \lambda \mu_B(x)$, so measures represent quantities in a given dimension in the same way that coordinates represent vectors.

4.4. Groups of dimensions; cardinality of bases. If X is commutative then X/\sim is also commutative, and in this section we prove that X/\sim is actually a free abelian group, not only a commutative monoid.

Proposition 9. *Let X be a quantity space over K . For every dimension $[x] \in X/\sim$ there is a unique dimension $[x]^{-1} \in X/\sim$ such that $[x][x]^{-1} = [x]^{-1}[x] = [\mathbf{1}]$.*

Proof. Let $x = \mu \prod_{i=1}^n b_i^{k_i}$ be the unique expansion of x relative to the basis $\{b_1, \dots, b_n\}$, and set $y = 1 \prod_{i=1}^n b_i^{k_i}$ and $z = 1 \prod_{i=1}^n b_i^{-k_i}$. (By Claim 2, y depends on $[x]$, but not on its representative x .) Then $[x] = [y]$ and $[y][z] = [z][y] = [zy] = [\mathbf{1}]$, so $[x][z] = [z][x] = [\mathbf{1}]$. Furthermore, if $[x]^{-1}[x] = \mathbf{1}$ then $[x]^{-1} = [x]^{-1}([x][z]) = ([x]^{-1}[x])[z] = [z]$. \square

Proposition 10. *Let X be a quantity space over K .*

- (1) *If $B = \{b_1, \dots, b_n\}$ is a basis for X , then $B^* = \{[b_1], \dots, [b_n]\}$ is a basis with the same cardinality for X/\sim .*
- (2) *Conversely, if $B^* = \{[b_1], \dots, [b_n]\}$, where each b_i is invertible, is a basis for X/\sim , then $B = \{b_1, \dots, b_n\}$ is a basis with the same cardinality for X .*

Proof. (1). The unique expansions of $b_i, b_{i'} \in B$ relative to B are $b_i = 1b_i$ and $b_{i'} = 1b_{i'}$, respectively, so $\mu_B(b_i) = \mu_B(b_{i'}) = 1$. Hence, $[b_i] = [b_{i'}]$ implies $b_i = b_{i'}$ according to Claim 2, so the mapping $b_i \mapsto [b_i]$ is one-to-one.

Let $[x]$ be an arbitrary dimension in X/\sim . As B generates X , $x = \mu \prod_{i=1}^n b_i^{k_i}$ for some $\mu \in K$ and some integers k_1, \dots, k_n , so $[x] = [\mu \prod_{i=1}^n b_i^{k_i}] = [\prod_{i=1}^n b_i^{k_i}] = \prod_{i=1}^n [b_i]^{k_i}$, so B^* generates X/\sim .

Also, if $[x] = \prod_{i=1}^n [b_i]^{k_i} = \prod_{i=1}^n [b_i]^{\ell_i}$, then $[\prod_{i=1}^n b_i^{k_i}] = [\prod_{i=1}^n b_i^{\ell_i}]$, so $k_i = \ell_i$ for $i = 1, \dots, n$ by Claim 2.

(2). If $b_i = b_{i'}$ then obviously $[b_i] = [b_{i'}]$, so the mapping $[b_i] \mapsto b_i$ is one-to-one.

Let x be an arbitrary quantity in X . We have $[x] = \prod_{i=1}^n [b_i]^{k_i} = [\prod_{i=1}^n b_i^{k_i}]$, and as $\prod_{i=1}^n b_i^{k_i}$ is invertible, Proposition 7 (1) implies that there exists some $\kappa \in K$ such that $x = \kappa \prod_{i=1}^n b_i^{k_i}$.

Finally, if $x = \mu \prod_{i=1}^n b_i^{k_i} = \nu \prod_{i=1}^n b_i^{\ell_i}$ then $[\mu \prod_{i=1}^n b_i^{k_i}] = [\nu \prod_{i=1}^n b_i^{\ell_i}]$, so $[\prod_{i=1}^n b_i^{k_i}] = [\prod_{i=1}^n b_i^{\ell_i}]$, so $\prod_{i=1}^n [b_i]^{k_i} = \prod_{i=1}^n [b_i]^{\ell_i}$, so $k_i = \ell_i$ for $i = 1, \dots, n$, since B^* is a basis for X/\sim . Also, $x = \mu y = \nu y$, where y is invertible, so $\mu = \nu$ by Proposition 7 (2). \square

Corollary 2. *Let X be a (finite-dimensional) quantity space over K .*

- (1) *X/\sim is a free abelian group (of finite rank).*
- (2) *Any two bases for X/\sim have the same number of elements, any basis for X has the same number of elements as any basis for X/\sim , and any two bases for X have the same number of elements.*

Proof. (1) is immediate. To prove (2), it suffices to note that any two bases for a free abelian group have the same cardinality, namely the rank of the group. \square

4.5. More general quantity spaces. Some physical quantities, such as a distance or a mass, can only have positive or non-negative measures. Applications involving such quantities require generalized quantity spaces. Specifically, the field acting on the monoid of quantities has to be replaced by a more general structure, as done in [Jon14a]. There, the concept of a scaloid over a field is replaced by the concept of a scaloid over a so-called *scalar system*. A scalar system can be conveniently defined as a subset of a field, inheriting addition and multiplication in the field, such that it is closed under addition and its non-zero elements constitute a group under multiplication. The real numbers, the non-negative real numbers, and the positive real number are obvious and important examples of scalar systems.

It is easy to verify that all propositions about quantity spaces in this section hold also for generalized quantity spaces over scalar systems, although null quantities require a semifield, and negative quantities require a field.

Remark. It is of course also possible to generalize quantity spaces over fields to free commutative scaloids over rings – or, perhaps, free “scalabel” monoids.

5. IN CONCLUSION

As shown in the Introduction, scalable monoids complement rings and modules from an abstract mathematical point of view. From the point of view of applications, a quantity space is a natural counterpart to a vector space: quantities and vectors are both fundamental notions in physics and quantitative sciences generally. Recall that the transformation of vector space theory into an axiomatic, “coordinate-free” form was completed during the interwar period in Europe, more than three quarters of a century ago, but the corresponding formulation of an axiomatic, “coordinate-free” notion of quantities has not yet been completed, in my opinion, despite important contributions (e.g., [Dro53, Whi68a, Whi68b]). Indeed, the research reported here started as an attempt to model quantities, not an attempt to fill a mathematical lacuna.

A crucial feature of systems of quantities is that two quantities can be added if and only if they are “commensurable”, so mathematical models of systems of quantities should reflect this peculiar property, not shared by numbers without sorts. An algebraic structure satisfying this requirement can of course be found if sufficiently complicated algebraic structures are considered. However, it would seem to be desirable that systems of quantities be modeled by a formal mathematical structure defined in a simple, direct way, similar to the definition of a vector space, in such a way that the crucial “commensurability” feature is a natural consequence of this formulation. The definitions given here would seem to pass this test.

Applications of quantity spaces, considered only informally and superficially here, are discussed at some length in [Jon14a, Jon14b].

REFERENCES

- [Bar96] Barenblatt G.I. (1996). *Scaling, Self-similarity and Intermediate Asymptotics*. Cambridge University Press.
- [Bri22] Bridgman P.W. (1922). *Dimensional Analysis*. Yale University Press.
- [Dro53] Drobot S. (1953). “On The Foundations of Dimensional Analysis”. *Studia Mathematica*, **14**, 84–99.
- [Jon14a] Jonsson D. (2014). Quantities, Dimensions and Dimensional Analysis. arXiv:1408.5024.
- [Jon14b] Jonsson D. (2014). Dimensional Analysis: A Centenary Update. arXiv:1411.2798.
- [Whi68a] Whitney H. (1968). “The Mathematics of Physical Quantities. Part I: Mathematical Models for Measurement”, *American Mathematical Monthly*, **75**, 115–138.
- [Whi68b] Whitney H. (1968). “The Mathematics of Physical Quantities. Part II: Quantity Structures and Dimensional Analysis”, *American Mathematical Monthly*, **75**, 227–256

UNIVERSITY OF GOTHENBURG, SE-405 30 GOTHENBURG, SWEDEN
E-mail address: dan.jonsson@gu.se